UNIVERSITY OF PLYMOUTH

Preparation Work

The best way to improve your understanding of mathematics in preparation for the degree is to solve problems. Below are some questions to help you. Please bring your solutions to the seven questions marked with a * to induction week when you can go through your work with your personal tutor. You can, of course, also then discuss any questions you have queries about.

- Calculators should not be used (except afterwards to check your work).
- Show all workings in a legible manner.
- Mathematical presentations are in English, so follow the rules of English.

Note that your results for many questions can be checked using the Wolfram alpha web site, (http://www.wolframalpha.com).

It will be useful for you to read The Code Book by Simon Singh before your arrival as the mathematics of cryptography will be part of the start of your course.

1 Algebra and Functions Examples

Strengthening your algebra skills is one of the best things you can do to ensure your success on a mathematics degree. Here are some questions to help you.

Question 1.* If

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f},$$

find u in terms of v and f.
Are you confident with your answer? Also does your result look elegant, can you simplify your expression? You can substitute some values to check it.

Question 2. Rearrange

$$\frac{x+1}{x-1} = 2,$$

to find x.
Comment: it is easy to check your result by substituting it back in. This is good practice.

Question 3. Similarly if

$$y = \frac{2x-3}{3x+2},$$

find x in terms of y.

Question 4. Factorise

$$x^4 - 1,$$

as much as you can.

Question 5. Find all the values of x that solve

$$\frac{x^2-2x}{x^2+1}=0\,.$$

Question 6. What are the exact numerical values of (revision link)

$$\log_{10}(100)\,;\quad \log_2(8)\,;\quad \log_3(27)\,;\quad \log_2(0.25)\,;\quad \log_{1234}(1)\,.$$

Question 7. Combine the following into one logarithm

$$\ln|x^2-1|+2\ln|\sqrt{x^2+1}|\,.$$

Does your result look elegant?
Note that $\ln|x|=\log_e|x|$ is the logarithm to the base $e\approx 2.718281828459\ldots$
We write the modulus of the argument as the argument of a logarithm must be positive for it to be real valued.

Question 8. If $f(x)=x^2$, show that

$$f(x+\delta)-f(x)=2x\delta+\delta^2\,.$$

Question 9. * If $f(x)=x^2$ and $g(x)=(x+1)^2$, show that the difference of the following compositions is (revision link)

$$f(g(x))-g(f(x))=4x(x^2+x+1)\,.$$

2 Differentiation Examples

The two key rules of differentiation are the product and chain rules. (The quotient rule can be derived from the other two.) The best way to write the product rule is

$$\frac{d}{dx}\left(f(x)g(x)\right)=f'(x)g(x)+f(x)g'(x)\,,$$

where we have maintained the order of f and g. This may be extended to products of more than two functions such as

$$\frac{d}{dx}\left(f(x)g(x)h(x)\right)=f'(x)g(x)h(x)+f(x)g'(x)h(x)+f(x)g(x)h'(x)\,.$$

The chain rule (and its special case, the chain rule for powers) are best presented through examples (revision link). Consider

$$\frac{d}{dx}\sin(3x^2)\,,$$

where we have to differentiate a function of a function, i.e., we require the chain rule. We write

$$\frac{d}{dx}\sin(3x^2) = \cos(3x^2)\frac{d}{dx}\left(3x^2\right) ,$$

where we have differentiated the function on the outside but not changed its argument and multiplied by the derivative of the argument. Carrying out this second differentiation generates our final result

$$\frac{d}{dx}\sin(3x^2) = 6x\cos(3x^2) .$$

Similarly to differentiate

$$\frac{d}{dx}\cos^4(3x^2) ,$$

we recognise that this notation means

$$\frac{d}{dx}\left(\left(\cos(3x^2)\right)^4\right) ,$$

so we have to differentiate a function raised to the power of 4. From the chain rule for powers

$$\frac{d}{dx}\left(\left(\cos(3x^2)\right)^4\right) = 4\left(\cos(3x^2)\right)^3\frac{d}{dx}\left(\cos(3x^2)\right) ,$$

i.e., we have pulled the power down, reduced the power by one and multiplied by the derivative of what was in the brackets. As in the first example, we use the chain rule and (check!) to obtain

$$\frac{d}{dx}\cos^4(3x^2) = -24x\sin(3x^2)\cos^3(3x^2) .$$

Question 1. Differentiate with respect to x

$$x\sin(5x) .$$

Question 2. Differentiate with respect to x

$$x\sin(5x)e^{-2x} .$$

Question 3. Differentiate with respect to x

$$x^2\cos(x^3) .$$

Question 4. Differentiate with respect to x

$$\sin^4(2x) .$$

Question 5.* Differentiate with respect to x

$$4\sin^4(5x^3) .$$

Some more exercises are available here: (revision link).

3 Integration by Parts Examples

Integration by parts is an important technique which you will use many times (revision link). Here is the key result which you should memorise

$$\int uv'\,dx = uv - \int u'v\,dx$$

Care is sometimes needed in choosing what part of the integrand should be identified with u and what with v'. The word LATE is often useful here. It stands for:

Logarithm**A**lgebra**T**rigonometry**E**xponential

and we choose u to be the structure which appears first in LATE. E.g., in

$$\int x^2 e^{-x}\,dx\,,$$

we would choose $u = x^2$ (Algebra) and $v' = e^{-2x}$ (Exponential) as A is before E in LATE.

Question 1. Use integration by parts to calculate the integral

$$\int xe^{-x}\,dx\,.$$

Check your result by differentiating it to demonstrate that you indeed recover xe^{-x}, the initial integrand.

Question 2.* If you want to integrate a function and do not see how, but you do know how to differentiate the function, a standard technique is to write the function as one times itself. Then you can use integration by parts and integrate the one. Use this approach to integrate the natural logarithm, i.e., calculate

$$\int \ln|x|\,dx = \int 1 \times \ln|x|\,dx\,.$$

where $\ln|x| = \log_e|x|$. Check your result by differentiating it and showing that you so recover the original integrand, $\ln|x|$.

4 Trigonometry Examples

Question 1. Two standard formulae are

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)\,,$$

and

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)\,.$$

Substitute these formulae into:

$$\sin^2(2\theta) + \cos^2(2\theta)\,,$$

and show, using $\sin^2(\theta) + \cos^2(\theta) = 1$, that you obtain one, as would, of course, be expected.

Question 2.* From the two standard formulae above and the definition of the tangent in terms of the sine and cosine functions, show that:

$$\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)} \,.$$

Question 3. Special angles whose trigonometric functions should be known are: $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$. The trigonometric functions of these angles can be derived from studying an equilateral triangle (drop a line from one vertex to the midpoint of the facing side, this gives a right angled triangle with angles $\pi/6$ and $\pi/3$) and a right angled isosceles triangle.
Derive the sine, cosine and tangent of these three important angles. These results can be easily checked (revision link).

5 Summation Notation Examples

Summation notation will be needed in a variety of contexts ranging from proofs in pure mathematics to solving problems in probability or statistics. Here are some exercises to refresh this notation (revision link).

Question 1.* Consider a set of four data points, x_i, where $x_1 = 2$, $x_2 = 3$, $x_3 = 1$, and $x_4 = -1$. Verify that

$$\frac{1}{4}\sum_{i=1}^{4} x_i = \frac{5}{4}\,, \quad \text{and} \quad \frac{1}{4}\sum_{i=1}^{4} x_i^2 = \frac{15}{4}\,.$$

Note the important consequence that $(\overline{x})^2 = 25/16$ (square of the average value) is not equal to $\overline{x^2} = 15/4$ (average of the squared values).

Question 2. Verify the results below for the following sums of counting numbers and the sum of their cubes

$$\sum_{i=1}^{4} i = 10\,, \quad \text{and} \quad \sum_{i=1}^{5} i^3 = 225\,.$$

Question 3. It will be proven in the first few weeks that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\,, \quad \text{and} \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}\,.$$

Thus write down

$$\sum_{i=1}^{10} i\,, \quad \text{and} \quad \sum_{i=1}^{10} i^2\,.$$

6 Binomial Coefficient Examples

The binomial coefficients are an efficient way to expand brackets and have many uses including combinatorics and Taylor series. Their definition in terms of factorials is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Recall also that 0! is taken to mean one.
You will already have seen these values written down in Pascal's triangle. Here are some exercises to prepare you for studying this in more depth on the course (revision link).

Question 1. Show that

$$\binom{n}{0} = \binom{n}{n} = 1\,,$$

and that

$$\binom{n}{1} = \binom{n}{n-1} = n\,,$$

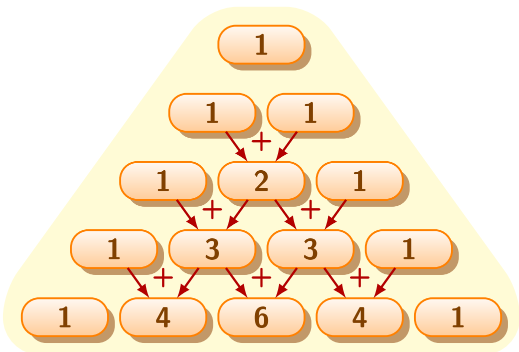
and that

$$\binom{n}{2} = \binom{n}{n-2} = \frac{n(n-1)}{2}\,.$$

Question 2.* Show the general symmetry property

$$\binom{n}{k} = \binom{n}{n-k}\,.$$

Question 3. In Pascal's triangle, the first few rows of which are shown below



the 1 at the top corresponds to the binomial coefficient $\binom{0}{0}$; the next row corresponds to $\binom{1}{0}$ and $\binom{1}{1}$. In turn the next row corresponds to $\binom{2}{0}$, $\binom{2}{1}$ and $\binom{2}{2}$ etc. Check that the binomial coefficients do indeed generate the numbers in the remaining rows.

Question 4. Demonstrate the additive method for obtaining the numbers in Pascal's triangle from the row above, i.e., show that for $k-1 \geq 0$ and $n-k \geq 0$

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}\,.$$